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Notes on new class for certain analytic functions

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1 Introducton

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$.

A function $f(z)\in\mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in\mathbb{U})$$

for some real number α with $0\leqq\alpha<1$. This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0)=\mathcal{S}^*$. The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [1]. It is well-known that $\mathcal{S}^*(\alpha)\subset\mathcal{S}^*\subset\mathcal{S}$.

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z)\prec q(z) \qquad (z\in\mathbb{U}), \tag{1.1}$$

if there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1 \quad (z\in\mathbb{U})$, and such that $p(z)=q(w(z)) \quad (z\in\mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

$$p(0)=q(0) \quad \text{and} \quad p(\mathbb{U})\subset q(\mathbb{U}). \tag{1.2}$$

Remark 1.1 Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. If $q(z)$ is univalent in $\mathbb{U}$, then the subordination (1.1) is equivalent to the condition (1.2).

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We define new class for certain analytic functions. Let $\mathcal{S}(\alpha,\beta)$ be the class of functions $f(z) \in \mathcal{A}$ which satisfy the inequality

$$\alpha < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real number α ($\alpha < 1$) and some real number β ($\beta > 1$).

Remark 1.2 Let $f(z) \in \mathcal{S}(\alpha,\beta)$. If $\alpha \geqq 0$, then $f(z)$ is starlike of order α in $\mathbb{U}$, which implies that $f(z)$ is univalent in $\mathbb{U}$.

Lemma 1.3 *Let $f(z) \in \mathcal{A}$. Then $f(z) \in \mathcal{S}(\alpha,\beta)$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) \qquad (z \in \mathbb{U}), \tag{1.4}$$

where $\alpha < 1$ and $\beta > 1$.

Proof. Let us consider the function $F(z)$ by

$$F(z) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) \qquad (z \in \mathbb{U}) \tag{1.5}$$

with $\alpha < 1$ and $\beta > 1$. Then, it is easy to see that the function $F(z)$ is analytic and univalent in $\mathbb{U}$ with $F(0) = 1$. Furthermore, noting that

$$1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) = \frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{\pi} i \log\left(\frac{ie^{-\pi i \frac{1-\alpha}{\beta-\alpha}} - ie^{\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right),$$

a simple check gives us that $F(z)$ maps $\mathbb{U}$ onto the strip domain w with $\alpha < \operatorname{Re} w < \beta$. Thus, it follows from Remark 1.1 that the subordination (1.4) is equivalent to the inequality (1.3), which proves the assertion of Lemma 1.3. □

We give some example for $f(z) \in \mathcal{S}(\alpha,\beta)$ as follows.

Example 1.4 Let us consider the function $f(z)$ given by

$$\begin{aligned} f(z) &= z \exp\left\{\frac{\beta-\alpha}{\pi} i \int_0^z \frac{1}{t} \log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} t}{1-t}\right) dt\right\} \\ &= z + \frac{\beta-\alpha}{\pi} i \left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right) z^2 + \cdots \qquad (z \in \mathbb{U}) \end{aligned} \tag{1.6}$$

with $\alpha < 1$ and $\beta > 1$. Then, we have

$$\frac{zf'(z)}{f(z)} = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) \qquad (z \in \mathbb{U}).$$

According to the proof of Lemma 1.3, it is clear that the function $f(z)$ given by (1.6) satisfies the inequality (1.3), which implies that $f(z) \in \mathcal{S}(\alpha,\beta)$.

2 Main results

Rogosinski [2] proved some coefficient estimates for subordinate functions.

Lemma 2.1 *Let* $q(z) = \sum_{n=1}^{\infty} B_n z^n$ *be analytic and univalent in* $\mathbb{U}$*, and suppose that* $q(z)$ *maps* $\mathbb{U}$ *onto a convex domain. If* $p(z) = \sum_{n=1}^{\infty} A_n z^n$ *is analytic in* $\mathbb{U}$ *and satisfies the following subordination*

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}),$$

then

$$|A_n| \leqq |B_1| \qquad (n = 1, 2, \cdots).$$

Applying Lemma 2.1, we deduced some coefficient estimates for $f(z) \in \mathcal{S}(\alpha, \beta)$ bellow.

Theorem 2.1 *If the function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}(\alpha, \beta)$*, then*

$$|a_n| \leqq \prod_{k=2}^{n} \frac{k - 2 + \dfrac{2(\beta - \alpha)}{\pi} \sin \dfrac{\pi(1-\alpha)}{\beta - \alpha}}{(n-1)!} \qquad (n = 2, 3, \cdots).$$

Proof. According to the assertion of Lemma 1.3, the function $f(z)$ satisfies the subordination (1.4). Let us define $p(z)$ and $q(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U}) \tag{2.1}$$

and

$$q(z) = 1 + \frac{\beta - \alpha}{\pi} i \log \left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta - \alpha}} z}{1 - z} \right) \qquad (z \in \mathbb{U}). \tag{2.2}$$

Then, the subordination (1.4) can be written as follows :

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}). \tag{2.3}$$

Note that the function $q(z)$ defined by (2.2) is convex in $\mathbb{U}$, and has the form

$$q(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta - \alpha}} \right) \qquad (n = 1, 2, \cdots).$$

If we let

$$p(z) = 1 + \sum_{n=1}^{\infty} A_n z^n,$$

then by Lemma 2.1, we see that the subordination (2.3) implies that

$$|A_n| \leqq |B_1| \qquad (n = 1, 2, \cdots), \tag{2.4}$$

where

$$|B_1| = \frac{\beta - \alpha}{\pi}\left|1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right| = \frac{2(\beta - \alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta - \alpha}. \tag{2.5}$$

Now, the equality (2.1) implies that

$$zf'(z) = p(z)f(z).$$

Then, the coefficients of z^n in both sides lead to

$$a_n = \frac{1}{n-1}\left(A_{n-1} + A_{n-2}a_2 + \cdots + A_1 a_{n-1}\right).$$

A simple calculation combined with the inequality (2.4) yields that

$$\begin{aligned}
|a_n| &= \frac{1}{n-1}\left|A_{n-1} + A_{n-2}a_2 + \cdots + A_1 a_{n-1}\right| \\
&\leqq \frac{1}{n-1}\left(|A_{n-1}| + |A_{n-2}||a_2| + \cdots + |A_1||a_{n-1}|\right) \\
&\leqq \frac{|B_1|}{n-1}\sum_{k=2}^{n}|a_{k-1}| \qquad (|a_1| = 1),
\end{aligned}$$

where B_1 is given in (2.5). To prove the assertion of the theorem, we need show that

$$|a_n| \leqq \frac{|B_1|}{n-1}\sum_{k=2}^{n}|a_{k-1}| \leqq \prod_{k=2}^{n}\frac{k-2+|B_1|}{(n-1)!}. \tag{2.6}$$

We now use the mathematical induction for the proof of the theorem.
Since

$$|a_2| \leqq |B_1||a_1| = |B_1|,$$

it is clear that the assertion is holds true for $n = 2$.
We assume that the proposition is true for $n = m$. Then, some calculation gives us that

$$\begin{aligned}
|a_{m+1}| &\leqq \frac{|B_1|}{(m+1)-1}\sum_{k=2}^{m+1}|a_{k-1}| = \frac{|B_1|}{m}\left(\sum_{k=2}^{m}|a_{k-1}| + |a_m|\right) \\
&\leqq \frac{|B_1|}{m}\left(1 + \frac{|B_1|}{m-1}\right)\sum_{k=2}^{m}|a_{k-1}| = \frac{m-1+|B_1|}{m}\,\frac{|B_1|}{m-1}\sum_{k=2}^{m}|a_{k-1}| \\
&\leqq \frac{m-1+|B_1|}{m}\prod_{k=2}^{m}\frac{k-2+|B_1|}{(m-1)!} = \prod_{k=2}^{m+1}\frac{k-2+|B_1|}{((m+1)-1)!},
\end{aligned}$$

which implies that the inequality (2.6) is true for $n = m + 1$.
By the mathematical induction, we prove that

$$|a_n| \leqq \prod_{k=2}^{n} \frac{k-2+|B_1|}{(n-1)!} \qquad (n = 2, 3, \cdots),$$

where B_1 is given in (2.5). This completes the proof of Theorem 2.2. □

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